*A Generalization of Volterra's Derivative of a Function of a Curve.**

By Charles Albert Fischer.

Volterra† has defined the derivative of a function of a curve at a point. He denotes the coordinates of the moving point which generates the curve by $(x, \phi(x))$, and the function and its derivative at the point $x = t$ by the expressions

$$y \,|\, [\phi(x)] \,|, \quad y' \,|\, [\phi(x), t] \,|,$$

respectively, and has proved the theorem that if the derivative is continuous and approached uniformly, then the first variation of the function is expressed by the equation

$$\delta y \,|\, [\phi(x)] \,| = \int_A^B y' \,|\, [\phi(x), t] \,| \,\delta \phi(t) \, dt.$$

He does not mention the order of approach to the derivative, but in the statement of his theorem he has assumed the derivative to be unique when the approach is of order zero.‡ In most of the applications it is necessary to assume an approach of higher order if the derivative is to be determined uniquely. Bliss, in an unpublished paper, has proved the same theorem in a slightly different form, introducing the idea of the order of approach to the derivative. He expresses the equations of the curve in parametric form, using the length of arc as the parameter, and assuming the variations to be orthogonal.

In the present paper we assume a set of m differential equations given, of the form

$$z_i' = g_i(x, y, y', z_1, z_2, \ldots, z_m) \qquad (i = 1, 2, \ldots, m),$$

where the functions $g_1, g_2, \ldots, g_m$ satisfy certain continuity conditions, and discuss the Volterra derivative of a function of the curve

$$C: \quad y = y(x) \qquad (x_1 \leqq x \leqq x_2)$$

relative to the set of curves along which the functions $z_1(x), z_2(x), \ldots, z_m(x)$,

* Read before the American Mathematical Society, February 22, 1913.

† Volterra: *Atti della R. Academia dei Lincei*, Ser. 4, Rendiconti III_2, p. 97.

‡ For the definition of the order of approach see § 3.

determined by the differential equations above, have given values at the end-points $x = x_1$ and $x = x_2$.

In § 1 two existence theorems are proved. The first has to do with the continuity of the solution of a set of ordinary differential equations, and the second is a modification of the existence theorem for implicit functions. In § 2 we define the set of curves to be considered and prove the existence of curves of the set having the properties required in the definition of the Volterra derivative. This section also contains a short discussion of the analogy between the theory developed in this paper and some simple theorems in the theory of functions of a finite number of variables. In § 3 will be found the definition of the Volterra derivative of a curve at a point, relative to the set of curves defined in § 2, and a proof of the theorem due to Volterra which has already been mentioned. The statement and proof of the theorem have necessarily been modified because of the special class of curves considered, but the theorem substantially as stated by Volterra falls out as a special case. The theorem as proved in § 3 will be called the generalized Volterra theorem. This section also contains the theorem that if the Volterra derivative of a function of a curve is continuous and approached uniformly with any finite order, then it must vanish identically along a curve which minimizes the value of the function considered. The last section deals with the application of the theory already developed to the Lagrange problem of the calculus of variations.

§ 1. *Auxiliary Theorems.*

We shall frequently consider systems of differential equations of the form

$$z_i' = g_i(x, \omega(x), z_1, z_2, \ldots, z_m) \qquad (i = 1, 2, \ldots, m),$$

where $\omega(x)$ represents a set of arbitrary functions which are always less in numerical value than a constant δ. The well-known existence theorem for the solution of a set of ordinary differential equations implies that if we assume a continuity of a high enough order for the functions g_i and ω there is a unique solution $z_1(x, \omega), z_2(x, \omega), \ldots, z_m(x, \omega)$ with a unique set of initial values and with continuity of any required order. It will also be necessary to know that the functions $z_i(x, \omega)$ approach the functions $z_i(x, 0)$ uniformly as $\boldsymbol{\omega}(x)$ approaches zero with δ. This will be established by means of the following theorem.

Suppose there is given a set of differential equations of the form

$$y_i' = \psi_i(t, y_1, y_2, \ldots, y_m;\ \eta_1, \eta_2, \ldots, \eta_r) = \psi_i(t, y; \eta) \qquad (i = 1, 2, \ldots, m;\ t_1 \leqq t \leqq t_2) \qquad (1)$$

containing the arbitrary functions $\eta_1, \eta_2, \ldots, \eta_r$ of t, and which for $\eta = 0$ have a solution

$$y_i = x_i(t) \qquad (i = 1, 2, \ldots, m;\ t_1 \leqq t \leqq t_2). \tag{2}$$

The functions ψ_i are supposed to be continuous and to have continuous first derivatives with respect to the variables y in a neighborhood R of the points $(t, x(t), 0)$ defined by the solutions (2). Let the absolute values of the functions η be restricted so as to satisfy the inequalities

$$|\eta_p(t)| \leqq \delta \qquad (p = 1, 2, \ldots, r;\ t_1 \leqq t \leqq t_2). \tag{3}$$

THEOREM I. *For every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever the functions η satisfy the inequalities* (3), *the solutions $y_i(t)$ of equations* (1) *which have the initial values*

$$y_i(t_1) = x_i(t_1) \qquad (i = 1, 2, \ldots, m)$$

are well defined over the entire interval $t_1 \leqq t \leqq t_2$ and satisfy the inequalities

$$|y_i(t) - x_i(t)| \leqq \varepsilon \qquad (i = 1, 2, \ldots, m;\ t_1 \leqq t \leqq t_2).$$

Let M/m be the largest of the maxima of the absolute values of the first partial derivatives of the functions $\psi_1, \psi_2, \ldots, \psi_m$ with respect to the variables y in the region R, and choose δ, ρ so that the region of points (t, y, η) defined by the inequalities

$$|\eta_p| \leqq \delta;\ x_i(t) - \frac{\rho}{M} e^{M(t_2 - t_1)} \leqq y_i \leqq x_i(t) + \frac{\rho}{M} e^{M(t_2 - t_1)};\ t_1 \leqq t \leqq t_2$$
$$(p = 1, 2, \ldots, r;\ i = 1, 2, \ldots, m)$$

lies entirely within R. The constant δ can furthermore be so restricted that the inequalities

$$|\psi_i(t, x(t); \eta) - \psi_i(t, x(t); 0)| \leqq \rho \qquad (t_1 \leqq t \leqq t_2)$$

are satisfied.

Define the functions $y_i^{(0)}(t), y_i^{(1)}(t), \ldots, y_i^{(n)}(t), \ldots$ by the equations

$$y_i^{(0)}(t) = x_i(t) \qquad (i = 1, 2, \ldots, m),$$

$$y_i^{(n)}(t) = x_i(t) + \int_{t_1}^{t} [\psi_i(t, y^{(n-1)}(t); \eta) - \psi_i(t, x(t); 0)]\, dt$$
$$(n = 1, 2, 3, \ldots).$$

Then, if the functions $y_i^{(n)}(t)$ approach definite limiting functions $y_i(t)$ as n becomes infinite, this set of functions $(y_1(t), y_2(t), \ldots, y_m(t))$ constitute a

solution of equations (1) having the required initial values.* It is evident that the equations

$$|y_i^{(1)}-y_i^{(0)}| = \left|\int_{t_1}^{t} \{\psi_i(t,x;\eta)-\psi_i(t,x;0)\}\,dt\right| \qquad (i=1,2,\ldots,m;\ t_1\leqq t\leqq t_2)$$

are satisfied. They imply the inequalities

$$|y_i^{(1)}-y_i^{(0)}| \leqq \rho\,(t-t_1).$$

Similarly,

$$|y_i^{(2)}-y_i^{(1)}| = \left|\int_{t}^{t_1} \{\psi_i(t,y^{(1)};\eta)-\psi_i(t,y^{(0)};\eta)\}\,dt\right|\cdot$$

Expanding the integrands by Taylor's formula and remembering the definition of the constant M, we find the inequalities

$$|y_i^{(2)}-y_i^{(1)}| \leqq \int_{t_1}^{t} M\rho\,(t-t_1)\,dt = M\rho\,\frac{(t-t_1)^2}{2\,!}\cdot$$

By a simple mathematical induction it can be proved that

$$|y_i^{(n)}-y_i^{(n-1)}| \leqq \frac{\rho}{M}\,\frac{M^n(t-t_1)^n}{n\,!} \qquad (i=1,2,\ldots,m;\ n=1,2,\ldots;\ t_1\leqq t\leqq t_2).$$

Hence the series

$$y_i^{(0)}+(y_i^{(1)}-y_i^{(0)})+\ldots+(y_i^{(n)}-y_i^{(n-1)})+\ldots$$

are uniformly convergent and their limits $y_i(t)$ satisfy the inequalities

$$|y_i(t)-x_i(t)| \leqq \frac{\rho}{M}\,e^{M(t_2-t_1)} \qquad (i=1,2,\ldots,m;\ t_1\leqq t\leqq t_2). \qquad (4)$$

If ρ is chosen so that the right member of the inequalities (4) is less than ε, the conditions of Theorem I are satisfied.

The existence theorem for implicit functions states that if there is given a solution

$$x=a;\ y_1=b_1,\ y_2=b_2,\ \ldots,\ y_m=b_m$$

of the algebraic equations

$$\phi_i(x;y_1,y_2,\ldots,y_m)=0 \qquad (i=1,2,\ldots,m),$$

where the functions $\phi_1, \phi_2, \ldots, \phi_m$ satisfy suitable continuity conditions, then for every value of x sufficiently near to a there is a unique set of values for $y_1, y_2, \ldots, y_m$ in the neighborhood of $b_1, b_2, \ldots, b_m$ which satisfy the equations. In the next section we shall have a similar situation, excepting that instead of a single variable x we shall have the infinity of variables represented

* This is the set of approximation functions usually used in Picard's method of approximation. See Bliss, *Annals of Mathematics,* Ser. 2, Vol. VI, p. 49. The proof that the functions defined by these sequences actually solve the differential equations, and the proof of the uniqueness of the solutions so determined, is the same as in the reference just given and is omitted here.

by a function $\eta(x)$ taken over the range $x_1 \leqq x \leqq x_2$. The introduction of the infinity of independent variables necessitates changes in the hypothesis and in the proof of the theorem. The proof given below is similar to a recent proof of the existence theorem for implicit functions due to Bliss.*

Consider a set of functions

$$\phi_i(H;\alpha_1,\alpha_2,\ldots,\alpha_m)=\phi_i(H;\alpha) \qquad (i=1,2,\ldots,m)$$

whose values are determined uniquely when the variables $\alpha_1,\alpha_2,\ldots,\alpha_m$ and a curve

$$H: \quad \eta=\eta(x) \qquad (x_1 \leqq x \leqq x_2)$$

in the $x\eta$-plane are given. The functions will be said to be continuous at the values $(\bar{H};\bar{\alpha}_1,\bar{\alpha}_2,\ldots,\bar{\alpha}_m)$, provided that for every positive constant e a second positive constant d can be found such that

$$|\phi_i(H;\alpha)-\phi_i(\bar{H};\bar{\alpha})| \leqq e$$

whenever

$$|\eta(x)-\bar{\eta}(x)|<d, \quad |\alpha_i-\bar{\alpha}_i| \leqq d \qquad (x_1 \leqq x \leqq x_2;\ i=1,2,\ldots,m).$$

Let the values $(H;\alpha_1,\alpha_2,\ldots,\alpha_m)=(H_0;0,0,\ldots,0)$, where H_0 is the curve $\eta(x)\equiv 0$, be a solution of the equations

$$\phi_i(H;\alpha)=0 \qquad (i=1,2,\ldots,m), \tag{5}$$

and suppose that, with respect to the variables α, the functions $\phi_1,\phi_2,\ldots,\phi_m$ are continuous and have continuous first partial derivatives in a domain

$$\eta(x) \leqq \varepsilon; \ |\alpha_i| \leqq \alpha^{(0)} \qquad (x_1 \leqq x \leqq x_2;\ i=1,2,\ldots,m). \tag{6}$$

At the initial values $(H_0;0)$ it will be supposed that these functions and their first partial derivatives are continuous with respect to all arguments in the sense described above, and furthermore that the functional determinant

$$\begin{vmatrix} \dfrac{\partial\phi_1(H_0;0)}{\partial\alpha_1} & \dfrac{\partial\phi_1(H_0;0)}{\partial\alpha_2} & \cdots & \dfrac{\partial\phi_1(H_0;0)}{\partial\alpha_m} \\ \dfrac{\partial\phi_2(H_0;0)}{\partial\alpha_1} & \dfrac{\partial\phi_2(H_0;0)}{\partial\alpha_2} & \cdots & \dfrac{\partial\phi_2(H_0;0)}{\partial\alpha_m} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial\phi_m(H_0;0)}{\partial\alpha_1} & \dfrac{\partial\phi_m(H_0;0)}{\partial\alpha_2} & \cdots & \dfrac{\partial\phi_m(H_0;0)}{\partial\alpha_m} \end{vmatrix} \tag{7}$$

is different from zero. Then the following theorem can be proved:

THEOREM II. *If $(H_0;0,0,\ldots,0)$ is an initial solution of the kind described above, then two positive constants ε and $\alpha^{(0)}$ can be determined such*

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* *Bulletin of the American Mathematical Society,* Vol. XVIII, No. 4, p. 175.

that no two solutions of equations (5) *in the domain* (6) *have the same argument* H. *Furthermore, a positive constant* $\delta \leqq \varepsilon$ *can be taken so small that for every curve* H *satisfying the inequality*

$$|\eta(x)| \leqq \delta \qquad (x_1 \leqq x \leqq x_2) \tag{8}$$

there exists a solution $(H; \alpha_1, \alpha_2, \ldots, \alpha_m)$ *of equations* (5) *in the domain* (6).

Choose the constants ε, $\alpha^{(0)}$ so small that in the domain (6) the inequality

$$\begin{vmatrix} \dfrac{\partial \phi_1(H;\alpha)}{\partial \alpha_1} & \dfrac{\partial \phi_1(H;\alpha)}{\partial \alpha_2} & \cdots & \dfrac{\partial \phi_1(H;\alpha)}{\partial \alpha_m} \\ \dfrac{\partial \phi_2(H;\alpha)}{\partial \alpha_1} & \dfrac{\partial \phi_2(H;\alpha)}{\partial \alpha_2} & \cdots & \dfrac{\partial \phi_2(H;\alpha)}{\partial \alpha_m} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial \phi_m(H;\alpha)}{\partial \alpha_1} & \dfrac{\partial \phi_m(H;\alpha)}{\partial \alpha_2} & \cdots & \dfrac{\partial \phi_m(H;\alpha)}{\partial \alpha_m} \end{vmatrix} \neq 0 \tag{9}$$

is always satisfied. The continuity of the functions $\phi_1, \phi_2, \ldots, \phi_m$ and their partial derivatives and the non-vanishing of the determinant (7) make such a choice of ε, $\alpha^{(0)}$ possible.

It will first be proved that there can not be two distinct solutions $(H; \alpha_1', \alpha_2', \ldots, \alpha_m')$ and $(H; \alpha_1, \alpha_2, \ldots, \alpha_m)$ of equations (5) in the domain (6). Suppose there are two such solutions. By means of Taylor's formula we can derive the equations

$$\phi_i(H;\alpha') - \phi_i(H;\alpha) = \sum_{j=1}^{m} \frac{\partial \phi_i}{\partial \alpha_j}(\alpha_j' - \alpha_j) = 0 \qquad (i = 1, 2, \ldots, m), \tag{10}$$

where the arguments of the partial derivatives of ϕ_i are

$$H; \alpha_1 + \theta(\alpha_1' - \alpha_1), \ldots, \alpha_m + \theta(\alpha_m' - \alpha_m),$$

in which $0 < \theta < 1$. The determinant of the coefficients of $\alpha_j' - \alpha_j$ in equations (10) cannot vanish on account of the inequality (9). Therefore the only solution of equations (10) is

$$\alpha_j' - \alpha_j = 0 \qquad (j = 1, 2, \ldots, m),$$

and the solutions $(H; \alpha')$, $(H; \alpha)$ of equations (5) are not distinct.

It will next be proved that a constant $\delta \leqq \varepsilon$ can be so restricted that, for every curve H satisfying the inequality (8), the equations (5) have a solution $(H; \alpha_1, \alpha_2, \ldots, \alpha_m)$ lying in the domain (6). Define a function Φ by the equation

$$\Phi(H; \alpha_1, \alpha_2, \ldots, \alpha_m) = \sum_{i=1}^{m} \phi_i^2(H; \alpha_1, \alpha_2, \ldots, \alpha_m).$$

The function $\Phi(H_0; \alpha)$ has a minimum for $\alpha_1 = \alpha_2 = \ldots = \alpha_m = 0$, since it vanishes for those values and is positive everywhere else in the region

$$|\alpha_i| \leqq \alpha^{(0)} \qquad (i = 1, 2, \ldots, m). \tag{11}$$

In particular, there is a positive number e such that the inequality

$$\Phi(H_0; \alpha) - \Phi(H_0; 0) > e$$

is satisfied for all points α on the boundary of the region (11). Then, if δ is sufficiently small, the inequality

$$\Phi(H; \alpha) - \Phi(H; 0) > e$$

is also satisfied for all functions $\eta(x)$ satisfying the condition (8) and all points α on the boundary of the region (11). The minimum of the function $\Phi(H; \alpha)$, considered as a function of $\alpha_1, \alpha_2, \ldots, \alpha_m$ alone, in the closed region (11) is therefore not at a boundary point of the region. The partial derivatives with respect to the variables α must vanish at the particular interior point of the region (11) at which $\Phi(H; \alpha)$ takes its minimum value. At this point we find the equations

$$\frac{1}{2}\frac{\partial \Phi}{\partial \alpha_1} = \phi_1 \frac{\partial \phi_1}{\partial \alpha_1} + \phi_2 \frac{\partial \phi_2}{\partial \alpha_1} + \ldots + \phi_m \frac{\partial \phi_m}{\partial \alpha_1} = 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$\frac{1}{2}\frac{\partial \Phi}{\partial \alpha_m} = \phi_1 \frac{\partial \phi_1}{\partial \alpha_m} + \phi_2 \frac{\partial \phi_2}{\partial \alpha_m} + \ldots + \phi_m \frac{\partial \phi_m}{\partial \alpha_m} = 0.$$

Since the determinant of the coefficients of $\phi_1, \phi_2, \ldots, \phi_m$ is distinct from zero, on account of the inequality (9), we know that equations (5) must be satisfied by the minimizing variables α. Thus the second part of the theorem is proved.

§ 2. *The Set of Curves K.*

Consider a set of differential equations

$$z_i' = g_i(x, y, y', z_1, z_2, \ldots, z_m). \tag{12}$$

Since the subscripts h, i, j always have the range $1, 2, 3, \ldots, m$ and the variable x the range $x_1 \leqq x \leqq x_2$ in this paper, we will hereafter not state the ranges explicitly in every set of equations. When a function $y = y(x)$ is given, equations (12) determine m functions $z_i(x)$ uniquely, if we assume the initial values z_{i1} of the functions $z_i(x)$ at $x = x_1$ to be given in advance. We are to consider the curves

$$C: \quad y = y(x) \tag{13}$$

joining two given points (x_1, y_1) and (x_2, y_2) in the xy-plane. *Such a curve*

is said to belong to the set of curves K if it passes through the points (x_1, y_1) and (x_2, y_2), and if furthermore, when $y(x)$ is substituted for y and its derivative for y' in equations (12), the corresponding functions $z_i(x)$ not only have the prescribed values z_{i1} at $x = x_1$ but also given values z_{i2} at $x = x_2$.

We will now take a particular curve C which belongs to the set K and which for the purposes of the following discussion will be assumed to be of class C'' * on the interval $x_1 \leqq x \leqq x_2$.

Associated with C there will be a set of functions

$$z_i = z_i(x) \tag{14}$$

having the given end-values z_{i1}, z_{i2} and satisfying equations (12). The functions $z_i(x)$ must be of class C'' in the discussion which is to follow. This continuity will be assured if in the $(m+3)$-dimensional space of points $(x, y, y', z_1, z_2, \ldots, z_m)$ in the neighborhood of the values defined by equations (13) and (14) the functions g_i are assumed to be of class C''.

There is a set of linear differential equations associated with equations (12), the so-called equations of variation,† which play an important rôle in the following pages. They are the equations

$$\zeta_i' = \frac{\partial g_i}{\partial y}\eta + \frac{\partial g_i}{\partial y'}\eta' + \sum_j \frac{\partial g_i}{\partial z_j}\zeta_j \tag{15}$$

which would be satisfied by the derivatives

$$\eta = \frac{\partial y(x,\alpha)}{\partial \alpha}, \quad \zeta_j = \frac{\partial z_j(x,\alpha)}{\partial \alpha}$$

of a family of systems of functions

$$y = y(x,\alpha), \quad z_i = z_i(x,\alpha)$$

satisfying equations (12), as is evident at once on differentiating those equations. The equations (15) are linear and non-homogeneous in $\zeta_1, \zeta_2, \ldots, \zeta_m$. If the first two terms of their right members are dropped, they become homogeneous, and there is consequently an adjoint system ‡

$$\lambda_i' + \sum_j \frac{\partial g_j}{\partial z_i}\lambda_j = 0, \tag{16}$$

which will also be of service.

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* Bolza: "Vorlesungen über Variationsrechnung," p. 14.

† Bolza: *loc. cit.*, p. 560, equations (58). Hadamard: "Leçons sur le Calcul des Variations," p. 226, equations (K).

‡ Hadamard: *loc. cit.*, pp. 33 and 230.

The curve C is said to be normal if there is no solution of equations (16) *which is also a solution of the equation*

$$\sum_j \left\{\lambda_j \frac{\partial g_j}{\partial y} - \frac{d}{dx}\left(\lambda_j \frac{\partial g_j}{\partial y'}\right)\right\} = 0, \tag{17}$$

where the arguments of the derivatives of g_j are x, $y(x)$, $y'(x)$, $z_1(x)$, $z_2(x)$, $\ldots$, $z_m(x)$, excepting the solution $(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_m) = (0, 0, 0, \ldots, 0)$.

The existence of such a solution of equations (16) and (17) is the first condition which must be satisfied if the curve C is to furnish a minimum for one of the functions, say $z_1(x)$, at $x = x_2$, when the prescribed values are given for all the functions $z_i(x)$ at $x = x_1$, and for all excepting $z_1(x)$ at $x = x_2$. The assumption that C is a normal curve is therefore equivalent to the assumption that C is not an extremal for the minimum problem just described.*

Let the systems $(\lambda_{h1}, \lambda_{h2}, \ldots, \lambda_{hm})$ be the solution of equations (16) which have the initial values †

$$\lambda_{ii}(x_2) = 1, \quad \lambda_{hi}(x_2) = 0 \qquad (i \neq h). \tag{18}$$

Any other solution will then be expressible in the form

$$\lambda_i(x) = \sum_h c_h \lambda_{hi}(x).$$

By means of the solutions $(\lambda_{h1}, \lambda_{h2}, \ldots, \lambda_{hm})$ of the adjoint equations (16), the values $\zeta_i(x_2)$ of the solutions of equations (15) which vanish at $x = x_1$ can readily be expressed in terms of the function $\eta(x)$. Multiply equations (15) by $\lambda_{hi}(x)$ and sum with respect to i. Then applying equations (16), we obtain the equations

$$\frac{d}{dx} \sum_i \lambda_{hi} \zeta_i = \sum_i \lambda_{hi} \left\{\frac{\partial g_i}{\partial y} \eta + \frac{\partial g_i}{\partial y'} \eta'\right\}. \tag{19}$$

If equations (19) are integrated between the limits x_1 and x_2, and the right members integrated by parts, they become

$$\zeta_h(x_2) = \int_{x_1}^{x_2} N_h(x)\, \eta(x)\, dx, \tag{20}$$

where

$$N_h(x) = \sum_i \left\{\lambda_{hi} \frac{\partial g_i}{\partial y} - \frac{d}{dx}\left(\lambda_{hi} \frac{\partial g_i}{\partial y'}\right)\right\}, \tag{21}$$

since it is always assumed that the function $\eta(x)$ vanishes at the end-points.

It will now be proved that, if C is a normal curve, a set of points ξ_1, ξ_2, $\ldots$, ξ_m, on the curve C, can always be selected such that the determinant

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* Bolza: *loc. cit.*, p. 564. Hadamard: *loc. cit.*, p. 234, equations (E).

† Hadamard: *loc. cit.*, p. 230–231.

$$\begin{vmatrix} N_1(\xi_1) & N_1(\xi_2) & \dots & N_1(\xi_m) \\ N_2(\xi_1) & N_2(\xi_2) & \dots & N_2(\xi_m) \\ \dots & \dots & \dots & \dots \\ N_m(\xi_1) & N_m(\xi_2) & \dots & N_m(\xi_m) \end{vmatrix} \neq 0. \tag{22}$$

It follows directly from the definition of a normal curve that ξ_1 can be taken so that $N_1(\xi_1) \neq 0$. Then the determinant

$$\begin{vmatrix} N_1(\xi_1) & N_1(x) \\ N_2(\xi_2) & N_2(x) \end{vmatrix}$$

can not vanish identically. Otherwise the expressions

$$N_1(\xi_1)\lambda_{2i} - N_2(\xi_1)\lambda_{1i}$$

would be a solution of equations (17) not identically zero, and the curve C would not be normal. Hence ξ_2 can be so chosen that it satisfies the equation

$$\begin{vmatrix} N_1(\xi_1) & N_1(\xi_2) \\ N_2(\xi_1) & N_2(\xi_2) \end{vmatrix} \neq 0.$$

Proceeding in this way with the determinant

$$\begin{vmatrix} N_1(\xi_1) & N_1(\xi_2) & N_1(x) \\ N_2(\xi_1) & N_2(\xi_2) & N_2(x) \\ N_3(\xi_1) & N_3(\xi_2) & N_3(x) \end{vmatrix}$$

and so on, it is easy to show that $\xi_1, \xi_2, \dots, \xi_m$ can be chosen so that the inequality (22) is satisfied.

Consider a constant ε, and any functions $\eta_1(x), \eta_2(x), \dots, \eta_m(x)$ such that $\eta_i(x)$ is of class C'' and vanishes identically everywhere excepting on the interval $\xi_i - \varepsilon < x < \xi_i + \varepsilon$, where it does not change its sign and is not identically zero, and it furthermore satisfies the inequalities

$$|\eta_i(x)| \leqq \varepsilon, \quad |\eta_i'(x)| \leqq \varepsilon, \quad |\eta_i''(x)| \leqq \varepsilon. \tag{23}$$

By applying the mean-value theorem for a definite integral to the determinant

$$\begin{vmatrix} \int_{\xi_1-\epsilon}^{\xi_1+\epsilon} N_1(x)\eta_1(x)\,dx & \int_{\xi_2-\epsilon}^{\xi_2+\epsilon} N_1(x)\eta_2(x)\,dx & \dots & \int_{\xi_m-\epsilon}^{\xi_m+\epsilon} N_1(x)\eta_m(x)\,dx \\ \dots & \dots & \dots & \dots \\ \int_{\xi_1-\epsilon}^{\xi_1+\epsilon} N_m(x)\eta_1(x)\,dx & \int_{\xi_2-\epsilon}^{\xi_2+\epsilon} N_m(x)\eta_2(x)\,dx & \dots & \int_{\xi_m-\epsilon}^{\xi_m+\epsilon} N_m(x)\eta_m(x)\,dx \end{vmatrix}$$

it can be reduced to the form

$$\begin{vmatrix} N_1(\xi_1^{(1)}) & N_1(\xi_2^{(1)}) & \dots & N_1(\xi_m^{(1)}) \\ \dots & \dots & \dots & \dots \\ N_m(\xi_1^{(m)}) & N_m(\xi_2^{(m)}) & \dots & N_m(\xi_{(m)}^{m}) \end{vmatrix} \prod_{i=1}^{m} \int_{\xi_i-\epsilon}^{\xi_i+\epsilon} \eta_i(x)\,ax,$$

where $\xi_i^{(j)}$ are certain mean values on the respective intervals $\xi_i - \varepsilon < x < \xi_i + \varepsilon$. Therefore, if ε is chosen sufficiently small, the inequality (22) implies the inequality

$$\begin{vmatrix} \int_{\xi_1-\epsilon}^{\xi_1+\epsilon} N_1(x)\eta_1(x)\,dx & \int_{\xi_2-\epsilon}^{\xi_2+\epsilon} N_1(x)\eta_2(x)\,dx & \dots\dots & \int_{\xi_m-\epsilon}^{\xi_m+\epsilon} N_1(x)\eta_m(x)\,dx \\ \dots & \dots & \dots & \dots \\ \int_{\xi_1-\epsilon}^{\xi_1+\epsilon} N_m(x)\eta_1(x)\,dx & \int_{\xi_2-\epsilon}^{\xi_2+\epsilon} N_m(x)\eta_2(x)\,dx & \dots\dots & \int_{\xi_m-\epsilon}^{\xi_m+\epsilon} N_m(x)\eta_m(x)\,dx \end{vmatrix} \neq 0, \quad (24)$$

since $N_h(x)$ is a continuous function of x.

If the points $\xi_1, \xi_2, \dots, \xi_m$ are considered as fixed and satisfying the conditions imposed above, it is possible to prove the following important theorem:

Suppose $\eta(x)$ is a function of class C'' which vanishes at the end-points $x=x_1$ and $x=x_2$ and satisfies the inequalities

$$|\eta(x)| \leqq \delta, \quad |\eta'(x)| \leqq \delta, \quad |\eta''(x)| \leqq \delta. \quad (25)$$

Suppose, further, that the functions $\eta_1(x), \eta_2(x), \dots, \eta_m(x)$ are such that $\eta_i(x)$ is of class C'', vanishes for all values of x excepting on the interval $\xi_i - \varepsilon < x < \xi_i + \varepsilon$, where it does not change its sign and is not identically zero, and satisfies the inequalities (23). *Then for $\varepsilon > 0$ it is possible to choose a positive constant $\delta \leqq \varepsilon$, such that, corresponding to every function $\eta(x)$, there is a set of functions $\eta_1(x), \eta_2(x), \dots, \eta_m(x)$ which make the curve*

$$C_\epsilon: \quad y = y(x) + \eta(x) + \sum_i \eta_i(x) \quad (26)$$

belong to the set K.

Choose an arbitrary set of functions $\bar\eta_1, \bar\eta_2, \dots, \bar\eta_m$ of class C'' which satisfy the inequalities (23) and vanish everywhere excepting on the specified intervals where they do not change their signs and are not identically zero, and let ε be chosen such that the inequality (24) is satisfied. Consider the curve

$$C_\epsilon: \quad y = y(x) + \eta(x) + \sum_i \alpha_i \bar\eta_i(x),$$

where the constants α_i are to be determined. Designate the solution of equations (12), when $y(x) + \eta(x) + \sum_i \alpha_i \bar\eta_i(x)$ has been substituted for y and its derivative for y', which has the given set of initial values $z_{11}, z_{21}, \dots, z_{m1}$ at $x = x_1$, by $z_1(x, \eta; \alpha), z_2(x, \eta; \alpha), \dots, z_m(x, \eta; \alpha)$, where α represents the set of parameters $\alpha_1, \alpha_2, \dots, \alpha_m$, and define $\phi_i(H; \alpha)$ by the equations

$$z_i(x_2, \eta; \alpha) - z_{i2} = \phi_i(H; \alpha).$$

The first theorem of § 1 implies that there is a constant $\varepsilon_0 > 0$, depending on

the magnitude of the region in which the functions g_i are continuous, such that, if we take $\varepsilon \leqq \varepsilon_0$ and $\alpha^{(0)}=1$, the expressions

$$\phi_i(H;\alpha),\quad \frac{\partial \phi_i(H;\alpha)}{\partial \alpha_j}$$

are continuous functions of the arguments α in the neighborhood defined by inequalities (23) and (6). They are also continuous at the values $(H_0; 0, 0, \ldots, 0)$ in the sense described in § 1, with respect to all arguments. Equations (20) imply the equations

$$\frac{\partial \phi_i(H_0;0)}{\partial \alpha_j} = \int_{\xi_j-\varepsilon}^{\xi_j+\varepsilon} N_i(x)\bar{\eta}_j(x)\,dx,$$

and the inequality (24) implies that the functional determinant

$$\begin{vmatrix} \dfrac{\partial \phi_1(H_0;0)}{\partial \alpha_1} & \dfrac{\partial \phi_1(H_0;0)}{\partial \alpha_2} & \cdots & \dfrac{\partial \phi_1(H_0;0)}{\partial \alpha_m} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial \phi_m(H_0;0)}{\partial \alpha_1} & \dfrac{\partial \phi_m(H_0;0)}{\partial \alpha_2} & \cdots & \dfrac{\partial \phi_m(H_0;0)}{\partial \alpha_m} \end{vmatrix}$$

is different from zero. Thus the hypothesis of the second theorem of § 1 is satisfied, and if δ is taken sufficiently small, there is a unique set of values for the parameters $\alpha_1, \alpha_2, \ldots, \alpha_m$, each less in absolute value than unity, corresponding to every curve H in the $x\eta$-plane which satisfies the condition (25), such that $(H; \alpha_1, \alpha_2, \ldots, \alpha_m)$ is a solution of the equations

$$\phi_i(H;\alpha) = z_i(x_2, \eta; \alpha) - z_{i2} = 0.$$

If we now let $\alpha_i\bar{\eta}_i = \eta_i$, the curve C_ε which is defined by equation (26) belongs to the set K. It is evident that if a constant δ is effective for a particular ε, say ε_0, it is also effective for every ε greater than ε_0.

The proofs in this section are unchanged if the functions considered are assumed to be of class C^r, $r>2$, instead of C'', and the inequalities (23) are replaced by the inequalities

$$|\eta_i(x)| \leqq \varepsilon,\quad |\eta_i'(x)| \leqq \varepsilon,\quad \ldots,\quad |\eta_i^{(r)}(x)| \leqq \varepsilon.$$

There is an interesting analogy between the results which have been obtained in this section, and some of the theorems in the theory of functions of a finite number of variables. The functions $z_i(x)$ which have the given initial values z_{i1} are uniquely determined by equations (12) when the function $y(x)$ is given, and therefore the quantities $z_i(x_2)$ are functions of the value of y at

every point on the range $x_1 \leqq x \leqq x_2$. If, instead of this infinity of variables $y(x)$ $(x_1 \leqq x \leqq x_2)$, we consider a finite set y_p $(p = 1, 2, \ldots, n)$, where the range $p = 1, 2, \ldots, n$ takes the place of the range $x_1 \leqq x \leqq x_2$, the equations $z_i(x_2) = z_{i2}$ have as their analogues a system of equations

$$z_i(y_1, y_2, \ldots, y_n) = z_{i2}. \tag{27}$$

Suppose there is given a set of values for $y_1, y_2, \ldots, y_n$ which satisfy equations (27), corresponding to the initial curve C of the set K. Suppose furthermore that an arbitrary increment η_p be given to each y_p, and that we wish to determine increments ω_{p_i} for m particular variables y_{p_i} such that the values

$$y_p + \eta_p \ (p \neq p_i), \quad y_{p_i} + \eta_{p_i} + \omega_{p_i} \tag{28}$$

shall satisfy equations (27). The condition that this is possible for every choice of η_p sufficiently small is the inequality

$$\begin{vmatrix} \dfrac{\partial z_1}{\partial y_{p_1}} & \dfrac{\partial z_1}{\partial y_{p_2}} & \cdots & \dfrac{\partial z_1}{\partial y_{p_m}} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial z_m}{\partial y_{p_1}} & \dfrac{\partial z_m}{\partial y_{p_2}} & \cdots & \dfrac{\partial z_m}{\partial y_{p_m}} \end{vmatrix} \neq 0, \tag{29}$$

which is the condition that equations (27) be solvable for the variables $y_{p_1}, y_{p_2}, \ldots, y_{p_m}$. If the determinant (29) vanishes for every choice of $p_1, p_2, \ldots, p_m$ there is a set of constants $\lambda_1, \lambda_2, \ldots, \lambda_m$, not all zero, which satisfy the equations

$$\sum_i \lambda_i \frac{\partial z_i}{\partial y_p} = 0 \qquad (p = 1, 2, \ldots, n).$$

This constitutes the first necessary condition that this set of values furnish a minimum for one of the functions, say $z_1(y_1, y_2, \ldots, y_m)$, with respect to those values for $y_1, y_2, \ldots, y_m$ which satisfy the equations

$$z_k(y_1, y_2, \ldots, y_n) = z_{k2} \qquad (k = 2, 3, \ldots, m). \tag{30}$$

Such a set of values for $y_1, y_2, \ldots, y_m$ is analogous to one of the curves C which is not normal according to the definition given above.

Suppose then that the set $(y_1, y_2, \ldots, y_n)$ satisfying equations (27) is normal in the sense that not all of the determinants of the matrix

$$\left\| \frac{\partial z_i}{\partial y_p} \right\| \qquad (i = 1, 2, \ldots, m; \ p = 1, 2, \ldots, n)$$

vanish, and suppose for convenience of notation that the determinant

$$\begin{vmatrix} \frac{\partial z_1}{\partial y_1} & \frac{\partial z_1}{\partial y_2} & \cdots & \frac{\partial z_1}{\partial y_m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial z_m}{\partial y_1} & \frac{\partial z_m}{\partial y_2} & \cdots & \frac{\partial z_m}{\partial y_m} \end{vmatrix}$$

is different from zero. Then we may take the set $(p_1, p_2, \ldots, p_m) = (1, 2, \ldots, m)$, a rearrangement which is not possible for the infinite set of variables $y(x)$, but which does not affect the argument when the number is finite. The theorem that for any set of increments $\eta_1, \eta_2, \ldots, \eta_n$ sufficiently small another set $\omega_1, \omega_2, \ldots, \omega_m$ can be determined, such that the values (28) satisfy equations (27), is analogous to the theorem proved above that the curve whose equation is

$$y = y(x) + \eta(x)$$

can always be made into a curve

$$C_\varepsilon: \quad y = y(x) + \eta(x) + \sum_i \eta_i(x)$$

of the set K by adding a properly chosen set of functions $\eta_i(x)$.

The derivative of another function $L(y_1, y_2, \ldots, y_n)$ with respect to one of the last n-m variables, say y_n, may be defined as the limit

$$\lim_{\eta_n=0} \frac{L(y_1+\omega_1, \ldots, y_m+\omega_m, y_{m+1}, \ldots, y_{n-1}, y_n+\eta_n) - L(y_1, y_2, \ldots, y_n)}{\eta_n}, \quad (31)$$

or with respect to one of the first m variables, say y_1, as

$$\lim_{\eta_1=0} \frac{L(y_1+\eta_1+\omega_1, y_2+\omega_2, \ldots, y_m+\omega_m, y_{m+1}, \ldots, y_n) - L(y_1, y_2, \ldots, y_n)}{\eta_1}. \quad (32)$$

In the former case the derivative has the value

$$\sum_{i=1}^{m} \frac{\partial L}{\partial y_i}\frac{\partial y_i}{\partial y_n} + \frac{\partial L}{\partial y_n},$$

where the factors $\partial y_i/\partial y_n$ are the derivatives found from the solutions of equations (27) for the variables $y_1, y_2, \ldots, y_m$. It must vanish if the function L is to have a minimum for the set of values $y_1, y_2, \ldots, y_n$. In the latter case the derivative is always zero, as can be easily verified. In the succeeding section we will define the derivative of an infinite number of variables $y(x)$ $(x_1 \leqq x \leqq x_2)$ in an analogous way, and prove that it must vanish for a minimum.

It should be noticed that in forming the derivatives (31) and (32) we divide by the arbitrary increment given to the variable with respect to which

we are differentiating, and do not divide by any of the increments which are given to the variables $y_1, y_2, \ldots, y_m$ in order that the restricting equations (27) may be satisfied.

§ 3. *The Derivative of a Function of a Curve.*

The Lagrange problem of the calculus of variations is to find the condition that a curve C shall minimize the value of a definite integral

$$\int_{x_1}^{x_2} f(x, y, y', z_1, z_1', \ldots, z_m, z_m')\, dx$$

taken over the set of curves K defined in the last section. The first necessary condition is the vanishing of an expression which we find to be the derivative of the definite integral considered as a function of the infinity of variables defining the curve C. In this section we will define the derivative for a much more general class of functions and find some of its properties, and in the next section we will discuss its application to the Lagrange problem.

If a number $L(C)$ corresponds to each curve C of a set K, then $L(C)$ is said to be a function of the curve C, as C ranges over K.

If a number $L(C, x)$ corresponds to each point x of each curve C of a set K, then $L(C, x)$ is said to be a function of the curve C at the point x.

If two curves

$$\begin{aligned} C: \quad & y = y(x) \qquad (x_1 \leqq x \leqq x_2), \\ \bar{C}: \quad & y = \bar{y}(x) \qquad (x_1 \leqq x \leqq x_2) \end{aligned}$$

are such that the inequalities

$$|\bar{y}(x) - y(x)| \leqq \varepsilon, \ |\bar{y}'(x) - y'(x)| \leqq \varepsilon, \ \ldots, \ |\bar{y}^{(r)}(x) - y^{(r)}(x)| \leqq \varepsilon \quad (x_1 \leqq x \leqq x_2)$$

are satisfied, then the curve $\bar{C}$ is said to be in the neighborhood $(\varepsilon)_r$ of the curve C.

Suppose C is a curve of class $C^{(r)}$, $r \geqq 2$, belonging to the set K. Take an arbitrary function $\eta(x)$ of class $C^{(r)}$ which vanishes identically everywhere excepting on the interval $\xi - \delta < x < \xi + \delta$, where it does not change its sign and is not identically zero, and which furthermore satisfies the inequalities

$$|\eta(x)| \leqq \delta, \quad |\eta'(x)| \leqq \delta, \quad |\eta''(x)| \leqq \delta.$$

Then choose a set of points $\xi_1, \xi_2, \ldots, \xi_m$ satisfying the conditions of the preceding section. According to the discussion there given, for a given ε the constant δ can be chosen such that a set of functions $\eta_1(x), \eta_2(x), \ldots, \eta_m(x)$, also satisfying the conditions of the preceding section, always exists, for which the curve

$$C_{\xi\epsilon}: \quad y = y(x) + \eta(x) + \sum_i \eta_i(x)$$

belongs to the set K.

If $L(C)$ is a function such that the limit

$$\lim_{\epsilon=0} \frac{L(C_{\xi\epsilon}) - L(C)}{\sigma} = L'(C, \xi; \xi_1, \xi_2, \ldots, \xi_m)$$

exists uniquely, where σ is defined by the equation

$$\sigma = \int_{\xi-\delta}^{\xi+\delta} \eta(x)\,dx,$$

then this limit is called the Volterra derivative of the function $L(C)$ at the point ξ, relative to the set of curves K and the set of points $\xi_1, \xi_2, \ldots, \xi_m$.

Under such circumstances, the Volterra derivative is said to be approached with order r.

If we form this derivative at a point of the set $\xi_1, \xi_2, \ldots, \xi_m$, say ξ_1, one admissible choice of the functions $\eta_i(x)$ is

$$\eta_1(x) = -\eta(x), \quad \eta_2(x) = \eta_3(x) = \ldots = \eta_m(x) = 0.$$

Then the curve $C_{\xi\epsilon}$ coincides with the curve C, and the Volterra derivative must vanish. Therefore, if the derivative exists at the points ξ_i, the equations

$$L'(C, \xi_i; \xi_1, \xi_2, \ldots, \xi_m) = 0$$

are satisfied.

When it will cause no ambiguity, we will let the expression $L'(C, \xi)$ represent the Volterra derivative, omitting the arguments $\xi_1, \xi_2, \ldots, \xi_m$.

We will always let C_ϵ represent a curve of class $C^{(r)}$ belonging to the set K, determined as in § 2. It is evident that every curve C_{ϵ_1} is also a curve C_ϵ, if $\epsilon_1 \leqq \epsilon$. $C_{\xi\epsilon}$ will always represent such a curve which also satisfies the conditions laid down in the definition of the Volterra derivative.

We assume that there is a neighborhood $(d)_r$ of the curve C which has the following property: For every $e > 0$ there exists an $\epsilon > 0$ such that if $\bar{C}$ and $\bar{C}_{\xi\epsilon}$ are curves of class $C^{(r)}$ belonging to the set K and in the neighborhood $(d)_r$ of C, and if ξ is any point on the interval $x_1 \leqq x \leqq x_2$, then the inequality

$$\left| \frac{L(\bar{C}_{\xi\epsilon}) - L(\bar{C})}{\sigma} - L'(\bar{C}, \xi) \right| \leqq e$$

is always satisfied. Under these circumstances the Volterra derivative is said to be approached uniformly with order r in the neighborhood $(d)_r$ of C.

We assume, further, that for every $e > 0$ there is an $\epsilon > 0$ such that if $\bar{C}$ is

any curve of class $C^{(r)}$ belonging to the set K in the neighborhood $(\varepsilon)_r$ of the curve C, then the inequality

$$|L'(\overline{C},x)-L'(C,x)|\leqq e$$

is satisfied. That is, the Volterra derivative is continuous with order r at the curve C, with respect to the set of curves K.

We assume, finally, that the Volterra derivative $L'(C,x)$ is a continuous function of x on the interval $x_1\leqq x\leqq x_2$, and since this interval is closed it is uniformly continuous.

Suppose $\eta(x,\alpha)$ is an arbitrary function of class $C^{(r)}$ in x which vanishes at the end-points $x=x_1$ and $x=x_2$ identically in α, and satisfies the following conditions. The functions $\eta(x,\alpha)$, $\eta'(x,\alpha)$, ..., $\eta^{(r)}(x,\alpha)$ all approach zero uniformly with respect to x when α approaches zero. The function $\eta(x,\alpha)$ has a derivative $\eta_\alpha(x,0)$ with respect to α at the point $\alpha=0$ which is continuous and approached uniformly with respect to x. If a small constant value is given to the parameter α, the function $\eta(x,\alpha)$ does not change its sign on the interval $x_1\leqq x\leqq x_2$.

It follows from these assumptions that when an $\varepsilon>0$ has been chosen it is always possible to restrict the value of α so that the functions $\eta(x,\alpha)$, $\eta'(x,\alpha)$, ..., $\eta^{(r)}(x,\alpha)$ will be less in absolute value than the δ determined in the preceding section, on the whole interval $x_1\leqq x\leqq x_2$. Then we can determine the functions $\eta_1(x,\alpha)$, $\eta_2(x,\alpha)$, ..., $\eta_m(x,\alpha)$ such that the curve

$$C_{\varepsilon a}:\quad y=y(x)+\eta(x,\alpha)+\sum_i\eta_i(x,\alpha)$$

shall belong to the set K, where $\eta_i(x,\alpha)$ satisfies the conditions for $\eta_i(x)$ in the preceding section.

The function $L(C_{\varepsilon a})$ is not uniquely determined by ε and α since the curve $C_{\varepsilon a}$ is not unique. Nevertheless the following theorem, which is referred to in the introduction as the generalized Volterra theorem, can be proved.

If the functions $L(C)$ and the curves C and $C_{\varepsilon a}$ satisfy the assumptions given above, then the expression

$$\lim_{\varepsilon=0}\frac{L(C_{\varepsilon a})-L(C)}{\alpha}$$

is unique, and its value is given by the equation

$$\lim_{\varepsilon=0}\frac{L(C_{\varepsilon a})-L(C)}{\alpha}=\int_{x_1}^{x_2}L'(C,x)\,\eta_\alpha(x,0)\,dx.$$

In particular, suppose $\eta(x,\alpha)$ is an admissible variation; that is, the curves

$$C_\alpha:\quad y=y(x)+\eta(x,\alpha)$$

belong to the set K. Then the functions $\eta_i(x,\alpha)$ are identically zero, and $L(C_\alpha)$ is a uniquely determined function of α, which has a derivative at $\alpha=0$ which is expressed by the equation

$$\left[\frac{dL(C_\alpha)}{d\alpha}\right]_{\alpha=0}=\int_{x_1}^{x_2}L'(C,x)\eta_\alpha(x,0)\,dx.$$

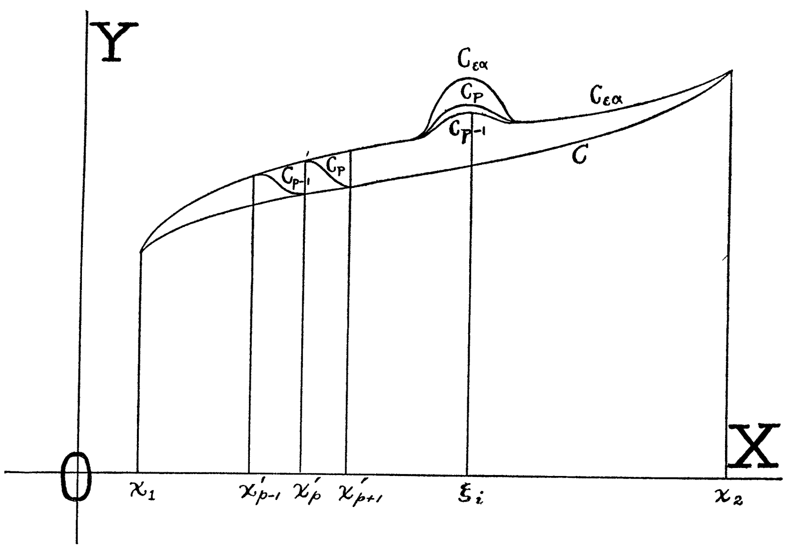
It should be noted that when ε approaches zero the parameter α approaches zero necessarily, and in such a way that every curve $C_{\varepsilon\alpha}$ belongs to the set K.

The theorem can be proved as follows: First choose an ε arbitrarily satisfying the conditions of § 2. Then determine δ, divide the interval $x_1\leqq x\leqq x_2$ into n segments each of length δ, and call the end-points of the segments $x_0', x_1', \ldots, x_n'$, so that $x_1=x_0'$ and $x_2=x_n'$. Select a set of functions $\theta_1(x)$, $\theta_2(x), \ldots, \theta_{n-2}(x)$ of class $C^{(r)}$ which satisfy the conditions

$$\begin{aligned}\theta_p(x)&=1 \qquad (x_1\leqq x\leqq x_p'),\\ 0<\theta_p(x)&<1 \qquad (x_p'<x<x_{p+1}'),\\ \theta_p(x)&=0 \qquad (x_{p+1}'\leqq x\leqq x_2),\end{aligned}$$

where p has the range $1, 2, \ldots, n-2$, and define $\theta_0(x)$ and $\theta_{n-1}(x)$ by the equations

$$\theta_0(x)\equiv 0,\quad \theta_{n-1}(x)\equiv 1.$$



It follows from the assumed properties of the function $\eta(x,\alpha)$ that for any choice of δ the value of α can be so restricted that the inequalities

$$|\theta_p(x)\eta(x,\alpha)|\leqq\delta,\quad \left|\frac{d}{dx}\{\theta_p(x)\eta(x,\alpha)\}\right|\leqq\delta,\quad \ldots,\quad \left|\frac{d^r}{dx^r}\{\theta_p(x)\eta(x,\alpha)\}\right|\leqq\delta$$

are satisfied on the ranges $x_1\leqq x\leqq x_2$ and $p=0, 1, 2, \ldots, n-1$.

Then, according to the principal theorem of § 2 we can choose the functions $\eta_{ip}(x,\alpha)$, where $i=1, 2, \ldots, m$ and $p=1, 2, \ldots, n-1$, in such a way that the curves

$$C_p:\quad y=y(x)+\theta_p(x)\eta(x,\alpha)+\sum_i \eta_{ip}(x,\alpha)$$

shall belong to the set K. It follows from the proof of that theorem that the function $\eta_{ip}(x,\alpha)$ can be made to differ from the function $\eta_{ip-1}(x,\alpha)$ only by a constant multiplier. Since the curve C_0 is the same as C, and C_{n-1} is the same as $C_{\varepsilon a}$, it is evident that

$$\frac{L(C_{\varepsilon a})-L(C)}{\alpha}=\sum_{p=1}^{n-1}\frac{L(C_p)-L(C_{p-1})}{\alpha}. \tag{33}$$

The curve C_p satisfies all the conditions imposed on the curve $C_{\xi\varepsilon}$ in the definition of the Volterra derivative when that derivative is taken for the curve C_{p-1} at the point x_p'. Since the Volterra derivative is assumed to be approached uniformly along every curve in a neighborhood $(d)_r$ of C, the equations

$$\frac{L(C_p)-L(C_{p-1})}{\alpha}=\frac{\sigma_p}{\alpha}[L'(C_{p-1},x_p')+\lambda_p] \qquad (p=1, 2, \ldots, n-1) \tag{34}$$

are satisfied, where the largest of the quantities $|\lambda_p|$ approaches zero with ε, and σ_p is defined by the equation

$$\sigma_p=\int_{x'_p}^{x'_{p+1}}\eta(x,\alpha)\,dx+\int_{x'_{p-1}}^{x'_p}(1-\theta_{p-1}(x))\eta(x,\alpha)\,dx-\int_{x'_p}^{x'_{p+1}}(1-\theta_p(x))\eta(x,\alpha)\,dx. \tag{35}$$

Since the derivative is continuous with respect to the set of curves K at the curve C, equations (34) may be written

$$\frac{L(C_p)-L(C_{p-1})}{\alpha}=\frac{\sigma_p}{\alpha}[L'(C,x_p')+\mu_p] \qquad (p=1, 2, \ldots, n-1), \tag{36}$$

where the largest of the quantities $|\mu_p|$ approaches zero with ε.

Since the ratio $\eta(x,\alpha)/\alpha$ approaches $\eta_\alpha(x,0)$ uniformly as α approaches zero, we know that the expression

$$\lim_{\varepsilon=0}\frac{1}{\alpha}\int_{x_1}^{x_2}\eta(x,\alpha)\,dx=\int_{x_1}^{x_2}\eta_\alpha(x,0)\,dx, \tag{37}$$

which is finite. The mean-value theorem reduces equations (35) to the equations

$$\begin{aligned}\sigma_p=\eta(x_p''',\alpha)(x_{p+1}'-x_p')+(1-\theta_{p-1}(x_{p-1}''))\eta(x_{p-1}'',\alpha)(x_p'-x_{p-1}')\\ -(1-\theta_p(x_p''))\eta(x_p'',\alpha)(x_{p+1}'-x_p'),\end{aligned} \tag{38}$$

where x_{p-1}'' is a point on the interval $x_{p-1}'<x<x_p'$, and x_p'' and x_p''' are points

on the interval $x'_p < x < x'_{p+1}$. It follows without difficulty from equations (37) and (38) that the equation

$$\lim_{\varepsilon=0} \sum_{p=1}^{n-1} \mu_p \frac{\sigma_p}{\alpha} = 0 \tag{39}$$

is satisfied. Since the Volterra derivative is a uniformly continuous function of x, the equations

$$\left.\begin{aligned} &\lim_{\varepsilon=0} \frac{1}{\alpha} \sum_{p=1}^{n-1} L'(C,x'_p)\,[\,(1-\theta_{p-1}(x''_{p-1}))\,\eta(x''_{p-1},\alpha)\,(x'_p-x'_{p-1}) \\ &\qquad\qquad -(1-\theta_p(x''_p))\,\eta(x''_p,\alpha)\,(x'_{p+1}-x'_p)\,] = \\ &\lim_{\varepsilon=0} \Big\{ \sum_{p=2}^{n-1} [L'(C,x_p)-L'(C,x'_{p-1})]\,(1-\theta_{p-1}(x''_{p-1}))\,\frac{\eta(x''_{p-1},\alpha)}{\alpha}\,(x'_p-x'_{p-1}) \\ &\qquad +L'(C,x'_1)\,(1-\theta_0(x''_0))\,\frac{\eta(x''_0,\alpha)}{\alpha}\,(x'_1-x'_0) \\ &\qquad -L'(C,x'_{n-1})\,(1-\theta_{n-1}(x''_{n-1})\,\frac{\eta(x''_{n-1},\alpha)}{\alpha}\,(x'_n-x'_{n-1}) \Big\} = 0 \end{aligned}\right\} \tag{40}$$

are easily verified.

If equations (36) and (38) are substituted in (33), the limit taken as ε approaches zero, and then equations (39) and (40) are subtracted, we have the equation

$$\lim_{\varepsilon=0} \frac{L(C_{\varepsilon\alpha})-L(C)}{\alpha} = \lim_{\varepsilon=0} \sum_{p=1}^{n-1} L'(C,x'_p)\,\frac{\eta(x'''_p,\alpha)}{\alpha}\,(x'_{p+1}-x'_p). \tag{41}$$

Since $\eta_\alpha(x,0)$ is approached uniformly and is continuous, we can state the equation

$$\frac{\eta(x'''_p,\alpha)}{\alpha} = \eta_\alpha(x'_p,0)+\nu_p, \tag{42}$$

where the largest of the quantities $|\nu_p|$ approaches zero with ε. If we now substitute equation (42) in (41), it easily reduces to the equation

$$\lim_{\varepsilon=0} \frac{L(C_{\varepsilon\alpha})-L(C)}{\alpha} = \int_{x_1}^{x_2} L'(C,x)\,\eta_\alpha(x,0)\,dx. \tag{43}$$

We will now consider the particular case where $\eta(x,\alpha)$ is a family of admissible variations. Since in the principal theorem of § 2 the multipliers $\alpha_1, \alpha_2, \ldots, \alpha_m$ were determined uniquely, and since in this case $\alpha_1=\alpha_2=\ldots=0$ is a correct determination of them, it is the only determination. Then the functions $\eta_i(x,\alpha)$ must vanish identically. Therefore when α is given, the curve C_α is determined uniquely, and equation (43) becomes

$$\left[\frac{dL(C_\alpha)}{d\alpha}\right]_{\alpha=0} = \int_{x_1}^{x_2} L'(C,x)\,\eta_\alpha(x,0)\,dx.$$

It follows from the generalized Volterra theorem and the fundamental lemma of the calculus of variations* that the identical vanishing of the Volterra derivative is a necessary condition that the curve C furnish a minimum for the value of the function $L(C)$ taken over the set of curves K, provided $L(C)$ is such a function that its Volterra derivative is continuous and approached uniformly with any finite order with respect to the set of curves K, and continuous also with respect to x, on the interval $x_1 \leqq x \leqq x_2$.

§ 4. *Applications to Functions Which Are Defined by Definite Integrals.*

In this section we will let the function $L(C)$ be a definite integral and find its Volterra derivative. It will appear that this derivative is the expression whose vanishing constitutes the first necessary condition for the curve C to minimize the value of the definite integral considered.

Define $L(C)$ by the equation

$$L(C) = \int_{x_1}^{x_2} f(x, y, y', z_1, z_1', \ldots, z_m, z_m')\, dx,$$

where the function f is assumed to be of class C'' in the $(2m+3)$-dimensional space of points $(x, y, y', z_1, \ldots, z_m')$ in the neighborhood of the values defined by the equations (13) and (14). The curve C is assumed to be a normal curve of class C''. Since the functions $f, g_1, g_2, \ldots, g_m$, together with their first and second partial derivatives, are continuous in the neighborhood mentioned above, they are uniformly continuous in every closed region included in that neighborhood.

Choose once for all a set of points $\xi_1, \xi_2, \ldots, \xi_m$ satisfying the conditions of § 2 with respect to the curve C. Then select a function $\eta(x)$ of class C'' which vanishes everywhere excepting on the interval $\xi - \delta < x < \xi + \delta$, where it does not change its sign and is not identically zero, and which satisfies inequalities (25). If ε is given, the value of δ can be restricted so that a set of functions $\eta_1(x), \eta_2(x), \ldots, \eta_m(x)$ satisfying the conditions of § 2 can be found, for which the curve

$$C_{\xi\varepsilon}: \quad y = y(x) + \omega(x)$$

belongs to the set K, where the function $\omega(x)$ is defined by the equation

$$\omega(x) = \eta(x) + \sum_i \eta_i(x).$$

We must compute the Volterra derivative from the equation

$$L'(C, \xi) = \lim_{\varepsilon = 0} \frac{L(C_{\xi\varepsilon}) - L(C)}{\sigma},$$

* Bolza: *loc. cit.*, p. 25.

where as usual

$$\sigma = \int_{\xi-\delta}^{\xi+\delta} \eta(x)\, dx.$$

Let $\zeta_i(x)$ represent the difference between the value of the function $z_i(x)$ taken along the curve $C_{\xi\varepsilon}$ and its value taken along the curve C. The expression $L(C_{\xi\varepsilon}) - L(C)$ can be expanded by Taylor's formula,* giving the equation

$$L(C_{\xi\varepsilon}) - L(C) = \int_0^1 \int_{x_1}^{x_2} \Big\{ f_y \omega + f_{y'} \omega' + \sum_j (f_{z_j} \zeta_j + f_{z'_j} \zeta'_j) \Big\} dx\, du, \qquad (44)$$

where the arguments of the partial derivatives of f are x, $y + u\omega$, $y' + u\omega'$, $z_1 + u\zeta_1, \ldots, z'_m + u\zeta'_m$. Since the integrand is uniformly continuous in both of the arguments x and u, we can change the order of integration whenever it is convenient. It follows immediately from equations (12) that the equations

$$\zeta'_i - [g_i(x, y + \omega, y' + \omega', \ldots, z_m + \zeta_m) - g_i(x, y, \ldots, z_m)] = 0$$

are satisfied for all values of x. Taylor's formula reduces them to the equations

$$\zeta'_i(x) - \int_0^1 \Big\{ \frac{\partial g_i}{\partial y} \omega + \frac{\partial g_i}{\partial y'} \omega' + \Sigma \frac{\partial g_i}{\partial z_j} \zeta_j \Big\} du, \qquad (45)$$

where the arguments of the partial derivatives of g_i are $x, y + u\omega, \ldots, z_m + u\zeta_m$. If equations (45) are multiplied by the undetermined functions $\lambda_i(x)$ of class C', then summed with respect to i, integrated between the limits x_1 and x_2, and added to equation (44), they give us the equation

$$L(C_{\xi\varepsilon}) - L(C) = \int_{x_1}^{x_2} \int_0^1 \Big\{ F_y \omega + F_{y'} \omega' + \sum_j (F_{z_j} \zeta_j + F_{z'_j} \zeta'_j) \Big\} du\, dx, \qquad (46)$$

where the function F is defined by the equation

$$F = f + \sum_i \lambda_i (z'_i - g_i), \qquad (47)$$

with the arguments the same as in equations (44) and (45).

Since all the functions in the integrand of equation (46) are uniformly continuous, the terms containing ω' and ζ'_j can be integrated by parts. The functions $\omega(x), \zeta_1(x), \ldots, \zeta_m(x)$ all vanish at both end-points, and equation (46) becomes

$$L(C_{\xi\varepsilon}) - L(C) = \int_{x_1}^{x_2} \Big\{ \omega \int_0^1 (F_y - F'_{y'})\, du + \sum_j \zeta_j \int_0^1 (F_{z_j} - F'_{z'_j})\, du \Big\} dx, \qquad (48)$$

where the accent always signifies the total derivative with respect to x. If the order of integration is changed in the first part of the right member of equation

* Jordan: "Cours D'Analyse," Vol. I, p. 247, equation (2).

(48), the function $\omega(x)$ replaced by $\eta(x)+\sum_j \eta_i(x)$, and the mean-value theorem applied, it reduces to the equation

$$L(C_{\xi\varepsilon})-L(C)=\sigma\int_0^1 (F_y-F'_{y'})_{x=\xi'}\,du+\sum_j \sigma_j\int_0^1 (F_y-F'_{y'})_{x=\xi'_j}\,du$$
$$+\sum_j\int_{x_1}^{x_2}\zeta_j\int_0^1 (F_{z_j}-F'_{z'_j})\,du\,dx, \tag{49}$$

where the points ξ' and ξ'_j are on the intervals $\xi-\delta<x<\xi+\delta$ and $\xi_j-\varepsilon<x<\xi_j+\varepsilon$ respectively, and the functions σ_j are defined by the equations

$$\sigma_j=\int_{\xi_j-\epsilon}^{\xi_j+\epsilon}\eta_j(x)\,dx.$$

We will determine the functions $\lambda_1(x), \lambda_2(x), \ldots, \lambda_m(x)$ by the differential equations

$$\int_0^1 (F_{z_j}-F'_{z'_j})\,du=0,$$

which may be written in the form

$$\lambda'_j+\sum_i \lambda_i\int_0^1 \frac{\partial g_i}{\partial z_j}\,du=\int_0^1 (f_{z_j}-f'_{z'_j})\,du. \tag{50}$$

The constants of integration will be determined by the equations

$$\int_0^1 \{F_y-F'_{y'}\}_{x=\xi'_j}\,du=0,$$

which may be written in the form

$$\sum_i\left\{\lambda_i(x)\int_0^1\frac{\partial g_i}{\partial y}\,du-\frac{d}{dx}\left(\lambda_i\int_0^1\frac{\partial g_i}{\partial y'}\,du\right)\right\}_{x=\xi'_j}=\int_0^1 (f_y-f'_{y'})_{x=\xi'_j}\,du. \tag{51}$$

This reduces equation (49) to the form

$$L(C_{\xi\varepsilon})-L(C)=\sigma\int_0^1 (F_y-F'_{y'})_{x=\xi'}\,du. \tag{52}$$

The existence theorem for a solution of a set of ordinary differential equations implies that the functions $\lambda_i(x)$ are of class C', if we can show that the constants of integration are finite and determined uniquely. Consider a particular solution $(\mu_1(x), \mu_2(x), \ldots, \mu_m(x))$ of equations (50) having an arbitrarily fixed set of initial values $\mu_{12}, \mu_{22}, \ldots, \mu_{m2}$ at $x=x_2$, and the fundamental system $(\lambda_{h1}, \lambda_{h2}, \ldots, \lambda_{hm})$ of solutions of the homogeneous equations

$$\lambda'_j+\sum_i \lambda_i\int_0^1\frac{\partial g_i}{\partial z_j}\,du=0, \tag{53}$$

with the initial values

$$\lambda_{hh}(x_2)=1,\quad \lambda_{hj}(x)=0 \qquad (h\neq j). \tag{18}$$

The functions μ_i and λ_{hi} depend on ω, but, according to the first theorem of § 1,

as ω approaches zero with ε, they approach as limiting values the corresponding solutions of the systems of equations (56) and (16) which are derived from (50) and (53) by putting $\omega=0$, $\varepsilon=0$. The general solution of equations (50) is expressed by the equation

$$\lambda_j(x) = \mu_j(x) + \sum_h c_h \lambda_{hj}(x). \tag{54}$$

The constants of integration c_h are determined by equations (51) which may be written in the form

$$\sum_h c_h \sum_i \left\{ \lambda_{hi} \int_0^1 \frac{\partial g_i}{\partial y} du - \frac{d}{dx}\left(\lambda_{hi} \int_0^1 \frac{\partial g_i}{\partial y'} du\right)\right\}_{x=\xi'_j} =$$
$$\left\{ \int_0^1 (f_y - f'_{y'}) du - \sum_i \left[\mu_i \int_0^1 \frac{\partial g_i}{\partial y} du - \frac{d}{dx}\left(\mu_i \int_0^1 \frac{\partial g_i}{\partial y'} du\right)\right]\right\}_{x=\xi'_j}$$

According to the first theorem of § 1 the determinant of the coefficients of the constants $c_1, c_2, \ldots, c_m$ reduces uniformly to the determinant (22) as ε approaches zero. Hence for a sufficiently small value of ε it is distinct from zero, and the constants $c_1, c_1, \ldots, c_m$ are determined uniquely. Evidently as ε approaches zero the constants themselves approach definite limiting values as is indicated by the theorem of § 1 mentioned above. This makes the functions $\lambda_j(x)$ approach uniformly the functions obtained by putting $\varepsilon=0, \omega=0$ in the equations determining them, as ε approaches zero.

If we now divide equation (52) by σ and let ε approach zero, we find the Volterra derivative to be

$$L'(C, \xi) = [F_y(x, y, y', z_1, \ldots, z'_m) - F'_{y'}(x, y, y', z_1, \ldots, z'_m)]_{x=\xi}, \tag{55}$$

where the functions $\lambda_i(x)$ are the solutions of the equations

$$F_{z_j} - F'_{z'_j} = -\lambda'_j - \sum_i \frac{\partial g_i}{\partial g_j} \lambda_i + f_{z_j} - f'_{z'_j} = 0 \tag{56}$$

whose constants of integration are determined by the equation

$$[F_y - F'_{y'}]_{x=\xi_j} = \left[f_y - f'_{y'} - \sum_i \left\{\lambda_i \frac{\partial g_i}{\partial y} - \frac{d}{dx}\left(\lambda_i \frac{\partial g_i}{\partial y'}\right)\right\}\right]_{x=\xi_j} = 0. \tag{57}$$

The well-known existence theorem for the solution of a set of ordinary differential equations implies that the functions $\lambda_i(x)$ and their derivatives are continuous. Since all the other functions involved in $L'(C, x)$ are also continuous by hypothesis, it is evident that $L'(C, x)$ is a continuous function of x over the range $x_1 \leqq x \leqq x_2$. It remains to be proved that it is approached uniformly with respect to the set of curves K in the neighborhood of the curve C, and is continuous in the argument C at the initial curve under consideration.

The condition which must be satisfied by the set of points $\xi_1, \xi_2, \ldots, \xi_m$ is expressed by the inequality (22). If we replace the curve C by any other curve

$$\overline{C}: \quad y = \bar{y}(x)$$

belonging to the set K and in a neighborhood $(\varepsilon)_2$ of the curve C, and choose ε sufficiently small, the resulting change in each element of the determinant (22) can be made less than any previously assigned positive constant, as is easily proved by means of the first theorem of § 1. We can therefore take ε so small that this determinant is distinct from zero for every admissible curve in the neighborhood $(\varepsilon)_2$ of C. That is, the set of points $\xi_1, \xi_2, \ldots, \xi_m$ is effective for every curve $\overline{C}$ in that neighborhood.

Since the first and second partial derivatives of f and g_i are uniformly continuous in a closed region including the curve C, the argument of this section remains valid if $\bar{y}(x)$ is substituted for $y(x)$ everywhere. Furthermore it can be shown that the functions $\lambda_i(x)$ approach the limiting values found by putting ε and consequently $\omega(x)$ equal to zero in equations (50) and (51), uniformly with respect to the curves of the set K in the neigborhood $(\varepsilon)_2$ of C. This assures us that the Volterra derivative is approached uniformly in the neighborhood $(\varepsilon)_2$ of the curve C.

To show that the Volterra derivative is continuous with respect to the set of curves K at the curve C we need simply state that if the quantities

$$|\bar{y}(x) - y(x)|, \quad |\bar{y}'(x) - y'(x)|, \quad |\bar{y}''(x) - y''(x)| \qquad (x_1 \leqq x \leqq x_2)$$

are taken sufficiently small, the difference between the values of $\zeta_i(x)$ and $\lambda_i(x)$ for the curve $\overline{C}$ and their values for the curve C can be made arbitrarily small uniformly. Therefore the value of the expression

$$|L'(\overline{C}, x) - L'(C, x)| \qquad (x_1 \leqq x \leqq x_2)$$

can be made less than an arbitrarily small constant by restricting the value of ε properly.

The hypothesis of the generalized Volterra theorem is completely satisfied for $r = 2$, and therefore the Volterra derivative whose value we have just found must vanish identically if the curve C minimizes the value of the function $L(C)$.

The expression which we have just found for the Volterra derivative has the same form as the left member of Euler's equation for the Lagrange problem of the calculus of variations.* The differential equations which determine the functions $\lambda_1(x), \lambda_2(x), \ldots, \lambda_m(x)$ in that problem† are our equations (56).

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* Bolza: *loc. cit.*, p. 563, equations (65).

† Bolza: *loc. cit.*, p. 562, equations (62).

The derivative is not determined uniquely by the arguments C, x since it is a function of the arguments $\xi_1, \xi_2, \ldots, \xi_m$ as well. The constants of integration involved in the functions $\lambda_i(x)$ were determined uniquely by the condition that the expression $F_y - F'_{y'}$ should vanish at the points $\xi_1, \xi_2, \ldots, \xi_m$, as expressed in equations (57). It is evident, then, that there is at most one set of constants which can make this expression vanish identically, as it must in case the curve C minimizes the value of the function $L(C)$. In that case the derivative is independent of the arguments $\xi_1, \xi_2, \ldots, \xi_m$. The constants of integration for the solution of equations (56) have been determined by different methods by Lagrange, Meyer, and perhaps others.* Since all of these methods must make the expression which I have called the Volterra derivative vanish identically along a minimizing curve, and in particular at the points $\xi_1, \xi_2, \ldots, \xi_m$, it is evident that the solutions of equations (56) which enter into the function F, in case of a minimum, must be the same, whatever method is used to determine the constants of integration.

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‡ Bolza: *loc. cit.*, pp. 563 and 566.